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The Lax pairs for the Holt system

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Abstract. By using the known relation between the Holt system and the Hénon–Heiles system, the Lax pairs for all the integrable cases of the Holt system are constructed from the known Lax representations for the Hénon–Heiles system.

1. Introduction

The Holt system is defined by the Hamilton function

$$\tilde{H} = \frac{1}{2}(p_X^2 + p_Y^2) + aX^{-2/3}\left(\frac{3b}{4}X^2 + Y^2 + c\right). \tag{1.1}$$

Only three integrable cases are known [1, 2]

$$\text{(i) } b = 1 \qquad \text{(ii) } b = 6 \qquad \text{(iii) } b = 16 \tag{1.2}$$

while the remaining parameters a and c are arbitrary constants. These parameters were isolated by the singular analysis [2], although the second integrals may be obtained directly [1, 3].

By integrability we mean the existence of a second independent integral of motion K, and in this case the Liouville theorem implies that the problem can be solved by quadratures. This, however, can be done only after finding special new variables which separate the associated Hamilton–Jacobi equation. Recall that for the Holt system the additional second integrals K are the polynomials of the third-, fourth- and sixth-order in momenta [1, 3], respectively. Therefore, it seems that the Hamiltonians (1.1) cannot be separable in the standard curvilinear coordinate systems. But at $b = 1, 6$ the Holt system belongs to the family of the Stäckel systems and the separation variables are related to the usual curvilinear coordinates. According to [4], at $b = 1, 6$ the second additional integral of motion K may be reduced to a quadratic polynomial in momenta $\{p_x, p_y\}$, which is related to the separability of the Hamilton–Jacobi equation in rotated Cartesian coordinates for (i) and in parabolic coordinates for (ii).

In fact, rescaling constants a and c in (1.1)

$$a \to 4\left(\frac{3}{2}\right)^{1/3} a \qquad c \to \frac{c}{3a}$$

and performing the canonical change of variables first proposed in [5]

$$X = \tfrac{2}{3}x^{3/2} \qquad p_X = p_x\sqrt{x}$$
$$Y = -\frac{1}{2\sqrt{3a}}p_y \qquad p_Y = 2\sqrt{3a}\,y$$

the Hamilton function (1.1) becomes

$$\tilde{H} = \frac{p_x^2 + p_y^2}{2x} + 2a(bx^2 + 3y^2) + \frac{2c}{x}. \tag{1.3}$$

According to [4], the time variable t and integrals of motion for the Holt system may be transformed by the rule

$$\begin{aligned} &\mathrm{d}\tilde{t} = x\,\mathrm{d}t \qquad \tilde{H} \mapsto H = x\tilde{H} \\ &\tilde{K} \mapsto K = \tilde{K} - \frac{y^n}{3}\tilde{H} \qquad n = [\sqrt{b}] = 1, 2, 4 \end{aligned} \tag{1.4}$$

into the integrals of motion for the Hénon–Heiles system

$$H = \frac{p_x^2 + p_y^2}{2} + 2ax(bx^2 + 3y^2) + 2c = T + V. \tag{1.5}$$

Here at $b = 1, 6$ the ratio of the Hamiltonians H and $\tilde{H}$ is equal to the ratio of the determinants of the associated Stäckel matrices [4]. This duality may be considered as the coupling-constant metamorphosis between integrable systems [6] with respect to the constant c in the potential V.

The purpose of this paper is to show how the canonical transformation of the extended phase space (1.4) acts on the Lax matrices $L(\lambda)$ and $A(\lambda)$ in the Lax equation

$$\frac{\mathrm{d}L(\lambda)}{\mathrm{d}t} = \{H, L(\lambda)\} = [A(\lambda), L(\lambda)] \tag{1.6}$$

and acts on the corresponding spectral curve

$$\mathcal{C}(z, \lambda) \qquad \det(zI + L(\lambda)) = 0. \tag{1.7}$$

The Lax representations for all the integrable cases of the Hénon–Heiles system were constructed in [8] by using the connection with stationary flows of some known integrable partial differential equations (PDEs). We shall use these Lax pairs to discuss the Lax representations for the Holt systems by exploiting transformation (1.4).

At $b = 6$ canonical transformation of the time (1.4) may be associated to the ambiguity of the Abel map on the corresponding hyperelliptic curve (1.7) [4]. The corresponding transformation of the Lax matrices and of the characteristic curve was considered in [4].

Below we shall consider two remaining cases at $b = 1$ and $b = 16$. Although the corresponding spectral curves are trigonal algebraic curves, transformations of the Lax matrices are similar to the transformations in the Kepler problem and in the case $b = 6$ [4].

In [7] the separability and another Lax pair for the Hénon–Heiles system have been considered. We shall use these results to construct a non-canonical transformation of the Hamiltonian (1.1) into the Stäckel form at $b = 16$ (iii) .

2. Results

Case (i). Let us begin with the Lax pair for the Hénon–Heiles system at $b = 1$ [8, 9]:

$$L(\lambda) = \begin{pmatrix} 6x\lambda & -\gamma(3x^2 + y^2) & \frac{3}{2\gamma}(3\lambda^2 - p_x) \\ \frac{3}{2\gamma}(3\lambda^2 + p_x) & -3x\lambda - \frac{p_y y}{\lambda} & \frac{y^2}{\lambda} \\ -\gamma(3x^2 + y^2) & 9\lambda^3 - \frac{y^2}{\lambda} & -3x\lambda + \frac{p_y y}{\lambda} \end{pmatrix}$$

and

$$A(\lambda) = \begin{pmatrix} 0 & 2\gamma\lambda & 0 \\ 0 & 0 & 1 \\ 2\gamma\lambda & -4\gamma^2 x & 0 \end{pmatrix} \qquad \text{where} \quad \gamma = \sqrt{3a}.$$

The corresponding spectral curve (1.7)

$$\mathcal{C}: \qquad -12az^3 = 729\lambda^7 - 162H\lambda^3 + 324c\lambda^3 + \frac{9K^2}{\lambda} \tag{2.1}$$

is a third-order cyclic covering of the line. Note that it is a very particular case in the class of generic trigonal algebraic curves.

Now we turn to the Holt system. As for the uniform Stäckel system [4], change of the time (1.4) induces the following transformations of the Lax matrices:

$$\tilde{L}(\lambda) = L(\lambda) + \frac{3}{2\gamma}H\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \tilde{A}(\lambda) = \frac{1}{x}A(\lambda).$$

The spectral curve of this matrix $L(\lambda)$ takes the following trigonal form:

$$\tilde{\mathcal{C}}: \qquad -12az^3 + 54\tilde{H}z\lambda^2 = 729\lambda^7 + 324c\lambda^3 + \frac{9\tilde{K}^2}{\lambda}.$$

Here K and $\tilde{K}$ are the second independent integrals of motion, which are second-order polynomials in momenta.

After the following point transformation:

$$x = \tfrac{1}{2}(u+v) \qquad y = \tfrac{1}{2}(u-v) \tag{2.2}$$

the integrals of motion for the Hénon–Heiles system are equal to

$$\begin{aligned} H &= p_u^2 + p_v^2 + a(u^3+v^3) + 2c \\ K &= p_u^2 - p_v^2 + a(u^3 - v^3). \end{aligned} \tag{2.3}$$

The same change of the variables for the Holt systems leads to

$$\begin{aligned} \tilde{H} &= 2\frac{p_u^2 + p_v^2 + a(u^3+v^3) + 2c}{u+v} \\ \tilde{K} &= 2\frac{v(p_u^2 + au^3 + c) - u(p_v^2 + av^3 + c)}{u+v}. \end{aligned} \tag{2.4}$$

For both systems u, v are separation variables and these systems belong to the Stäckel set of integrable systems [4].

In these separation variables, the Hénon–Heiles dynamics splits on two tori. According to [11], we can construct another 2×2 Lax representation for the Hénon–Heiles system with hyperelliptic spectral curve. Change of the time (1.4) induces transformation of the algebraic curves $\mathcal{C} \to \tilde{\mathcal{C}}$, which rearranges moduli H and $\tilde{H}$ and preserves the genus of the corresponding spectral curves. Therefore, by using a slightly different covering of the two tori the 2×2 Lax representation for the Holt system at $b = 1$ may be constructed as well.

Case (iii). At $b = 16$ the Lax matrices for the Hénon–Heiles system [8, 9] are

$$L(\lambda) = \begin{pmatrix} 12x - \frac{p_y y}{2\lambda} & \frac{y^2}{4\lambda} & \frac{3}{8a} \\ 9\lambda + 3p_x + \frac{6axy^2}{\lambda} - \frac{p_y^2}{2\lambda} & -6x & \frac{y^2}{4\lambda} \\ -24a\left(\frac{24x^2+y^2}{2} - \frac{xyp_y}{\lambda}\right) & 9\lambda - 3p_x - \frac{6axy^2}{\lambda} - \frac{p_y^2}{2\lambda} & -6x + \frac{p_y y}{2\lambda} \end{pmatrix}$$

and

$$A(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 24a(\lambda - p_x) & -48ax & 0 \end{pmatrix}.$$

As above, the spectral curve of the Lax matrix

$$\mathcal{C}: \qquad -8az^3 = 243\lambda^2 - 54H + 108c + \frac{3K^2}{\lambda^2}$$

is a third-order cyclic covering of the line.

Canonical change of the time (1.5) is closed to the Kepler transformation [4]. Hence, transformation of the Lax matrices and of the algebraic curves have a similar form:

$$\tilde{L}(\lambda) = L(\lambda) + 3H \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \tilde{A}(\lambda) = \frac{1}{x} A(\lambda) \tag{2.5}$$

and

$$\tilde{\mathcal{C}}: \qquad -8az^3 + 9\tilde{H}z = 243\lambda^2 + 108c + \frac{3\tilde{K}^2}{\lambda^2}.$$

Here the second integrals of motion $\tilde{K}$ and K are the non-factorable polynomial of the fourth order in momenta

$$K = \frac{p_y^4}{4} - 2y^6a^2 + 6axy^2{p_y}^2 - 2ap_y y^3 p_x - 12x^2y^4a^2 \tag{2.6}$$

and

$$\tilde{K} = \frac{p_y^4}{4} + \frac{1}{4}\frac{ay^2(y^2 + 24x^2){p_y}^2}{x} - 2ap_y y^3 p_x + \frac{1}{4}\frac{y^4a(4axy^2 + {p_x}^2 + 16ax^3 + 4c)}{x}. \tag{2.7}$$

According to [7, 10], let us introduce the Stäckel integrals of motion for the Hénon–Heiles system

$$\begin{aligned} H &= p_u^2 - 2au^3 + p_v^2 - 2av^3 + 2c \\ K &= p_u^2 - 2au^3 - p_v^2 + 2av^3 \end{aligned} \tag{2.8}$$

and for the Holt system

$$\begin{aligned} \tilde{H} &= -4\frac{{p_u}^2 - 2au^3 + {p_v}^2 - 2av^3 + 2c}{u+v} \\ \tilde{K} &= 2\frac{v({p_u}^2 - 2au^3 + c) - u({p_v}^2 - 2av^3 + c)}{u+v}. \end{aligned} \tag{2.9}$$

These integrals coincide with integrals (2.3), (2.4) up to the constant factors. Let us introduce the quasi-point transformation of variables [7, 10]

$$\begin{aligned} u &= \frac{1}{2}\frac{C}{y^2a} + \frac{1}{4}\frac{{p_y}^2}{y^2a} + x, & p_u &= \frac{1}{2}p_x + \frac{1}{2}\frac{p_y}{y}\left(-\frac{1}{2}\frac{{p_y}^2}{y^2a} - 6x - \frac{C}{y^2a}\right) \\ v &= -\frac{1}{2}\frac{C}{y^2a} + \frac{1}{4}\frac{{p_y}^2}{y^2a} + x & p_v &= \frac{1}{2}p_x + \frac{1}{2}\frac{p_y}{y}\left(-\frac{1}{2}\frac{{p_y}^2}{y^2a} - 6x + \frac{C}{y^2a}\right). \end{aligned} \tag{2.10}$$

Here constant of motion C is unspecified function of the new variables (x, p_x, y, p_y).

According to [12] we have to substitute the separation variables (2.10) into the definition of the second integrals K (2.8) and solve the resulting second-order equation $C = K$. Substituting the solution into (2.10) we get a change of variables, which transforms the Stäckel integrals (2.8) into the Hénon–Heiles integrals (1.5), (2.6) at $b = 16$. Moreover, this transformation is a canonical transformation of the variables [12].

For the Holt system we can also substitute new variables (2.10) into the definition of the corresponding second integrals $\tilde{K}$ (2.9) and solve the resulting second-order equation $C = \tilde{K}$. Substituting the solution into (2.10) one gets change of variables, which transforms the Stäckel integrals (2.9) into the Holt integrals (1.3), (2.7) at $b = 16$. In contrast to the Hénon–Heiles case, this change of variables is a non-canonical transformation.

So, at $b = 16$ we have a non-canonical change of variables

$$(t, x, y, p_x, p_y) \to (\tilde{t}, u, v, p_u, p_v)$$

which transforms integrals of motion for the Holt system into the Stäckel form. Of course, such transformations are known. As an example, the complete Kolosoff transformation [13] connects the Stäckel system with the Kowalewski top, which is an integrable but non-Stäckel system. By using such transformations we can construct the separated equations in the Lagrangian variables $(u, \dot{u}, v, \dot{v})$ and get solutions of the equations of motion in theta-functions. Up until now, in the quantum mechanics we have not been able to construct a counterpart of this transformation for the Holt system.

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